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ON THE RIEMANN-LEBESGUE LEMMA WITH A NONHARMONIC KERNEL

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Abstract

Proofs are provided to the Riemann-Lebesgue Lemma for general periodic and wavelet-like kernels and within the context of a recently advanced [6] nonharmonic form for this lemma, with a scale-frequency duplicated asymptotic parameter. We also introduce the concept of a symmetrized Riemann integral and demonstrate the existence of a Riemann-Lebesgue property for it.

1. The Riemann-Lebesgue Lemma and its Double Parameterization

The Riemann-Lebesgue Lemma RLL (see, e.g., [1], [8]) is widely recognized as one of the most important results in applied mathematics. A survey of the classical RLL can be found, e.g., in the famous monograph [8] by Zygmund as its Theorem 4.15 for a 2π- periodic kernel. The 2π- periodicity is, however, not a restriction since it is always possible to

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replace the $[0, 2\pi]$ interval by any $I \subset R = (-\infty, \infty)$ to restate the RLL as follows. Assume $f : I \mapsto R$ with $f \in L[I]$, Lebesgue integrable, and if $\mathfrak{g} : R \mapsto R$ is any measurable, bounded and I- periodic function, then

$$\lim_{\zeta \to \infty} \frac{1}{I} \int_I f(x)\ \mathfrak{g}(\zeta x)\ dx = \frac{1}{I} \int_I f(x)\ dx\ \frac{1}{I} \int_I \mathfrak{g}(x)\ dx,$$

where $\zeta = 2\pi / I$. Clearly, if $\mathfrak{g}$ satisfies the Bochner integrated symmetry condition

$$\frac{1}{I} \int_I \mathfrak{g}(x)\ dx = 0, \tag{1.1}$$

then

$$\lim_{\zeta \to \infty} \int_I f(x)\ \mathfrak{g}(\zeta x)\ dx = 0, \tag{1.2}$$

is an RLL in which the $\mathfrak{g}(\zeta x)$ kernel represents, restrictively, dilations of $\mathfrak{g}(x)$.

In a recent paper [6], the author introduced a generalized form for the Riemann-Lebesgue Lemma (RLL) on R by using a transform kernel composed of dilated trigonometric functions. A double parameterized kernel of this lemma has been employed in [6] to define a scale-frequency finite integral transform, which has much in common with the Gábor transform, towards a new scale-frequency series representation for a broad functional class covering periodic, almost periodic and aperiodic functions.

The first aim of this work is to show that the periodic kernel in the RLL (1.2) does not necessarily have to be dilational and can further be replaced by a more general periodic piece-wise continuous kernel $\mathfrak{g}_\zeta(x) \in L^2[I]$. Indeed, consider, without loss of generality, the example of an $I-$periodic waveform

$$\mathfrak{h}(x) = \sum_{n=-\infty}^{\infty} \{2u[x - (n + 1/2\ )I] - u[x - nI] - u[x - (n + 1)I]\}, \tag{1.3}$$

with $u(x)$ as Heaviside's unit step function, which satisfies (1.1). Its dilations

$$\mathfrak{h}(\zeta x) = \sum_{n=-\infty}^{\infty} \{2u[\zeta\, x - (n + 1/2\,)I] - u[\zeta\, x - nI] - u[\zeta\, x - (n+1)I]\}$$

can, on one hand, remain I- periodic only for integer scaling factors, i.e., when $\zeta = 2\pi / I = m$. On the other hand, an associated non-dilational kernel

$$\mathfrak{h}_\zeta(x) = \sum_{n=-\infty}^{\infty} \{2u[x - (n + 1/2\,)\frac{2\pi}{\zeta}\,] - u[x - n\,\frac{2\pi}{\zeta}\,] - u[x - (n+1)\frac{2\pi}{\zeta}\,]\} \neq \mathfrak{h}(\zeta x)$$

(1.4)

turns out to be l- periodic for an arbitrary $\zeta = 2\pi / l,$ including the case when $l = I.$

Motivated by this fact, we shall conceive in the forthcoming analysis the kernel $\mathfrak{g}_\zeta(x)$ as a certain family $(\mathfrak{g}_\zeta \mid \zeta \in J)$ of bounded measurable l- periodic functions $x \mapsto \mathfrak{g}_\zeta(x) \in L^2[l],$ the Hilbert space, with an index set $J \subset [0, \infty),$ that may be discrete or “continuous”, which satisfy a certain Bochner integrated symmetry condition $\forall\, \zeta \in J \subset [0, \infty).$ We can move, then to develop a concise short proof for the RLL that entertains such non-dilational kernels.

Lemma 1. *If* $\mathfrak{g}_\zeta(x)$ *is any measurable* l*-periodic piece-wise continuous function of* $L^2[l]$ *satisfying the Bochner integrated symmetry condition*

$$\frac{\zeta}{2\pi}\int_{-\frac{\pi}{\zeta}}^{\frac{\pi}{\zeta}} \mathfrak{g}_\zeta(x + \alpha)\; dx = 0, \quad \forall\, \zeta \in J \subset [0, \infty), \tag{1.5}$$

where $\alpha \in l$ *is at least one point where* $\mathfrak{g}_\zeta(x)$ *is uniformly continuous, and*

$$|\mathfrak{g}_\zeta(x)| < V(x) \in L[l],\ \forall\, \zeta \text{ and } \forall\, x \in l, \tag{1.6}$$

then

$$\lim_{\zeta \to \infty} \mathfrak{g}_\zeta(x) = \mathfrak{g}_\infty(x) \equiv 0. \tag{1.7}$$

Proof. Here l-periodicity, with $l = 2\pi / \zeta$, covers I as one of its instants. It is well known that the reversed version of the previous limit, which defines aperiodicity, is $\lim_{\zeta \to 0} \mathfrak{g}_\zeta(x) = \mathfrak{g}_0(x) = \frac{1}{2\pi} \int_R \widehat{\mathfrak{g}_0}\,(\omega)\, e^{i\omega x} d\omega$, where

$$\mathfrak{g}_0(x) \leftrightarrow \widehat{\mathfrak{g}_0}\,(\omega) = \int_R \mathfrak{g}_0(x) e^{-i\omega x} dx, \tag{1.8}$$

the corresponding Fourier transform pair. Alternatively, highly oscillatory periodic $\mathfrak{g}_\zeta(x) \in L^2[l]$ addressed in (1.7)- $L[l]$ is not enough due to the well known Kolmogorov's example are always expandable in Fourier series vis

$$\mathfrak{g}_\zeta(x) = \sum_{m=-\infty}^{\infty} \mathfrak{A}_\zeta(m)\; e^{im\,\zeta\,x},$$

with the coefficients

$$\mathfrak{A}_\zeta(m) = \frac{1}{l} \int_l \mathfrak{g}_\zeta(x) e^{-im\,\zeta\,x} dx = \frac{\zeta}{2\pi} \int_{-\frac{\pi}{\zeta}}^{\frac{\pi}{\zeta}} \mathfrak{g}_\zeta(x) e^{-im\,\zeta\,x} dx \tag{1.9}$$

always defined when the dominance condition (1.6), for which $\int_l |\mathfrak{V}| dx < \infty$, is satisfied. Moreover, it is graphically obvious that

$$\frac{\zeta}{2\pi} \int_{-\frac{\pi}{\zeta}}^{\frac{\pi}{\zeta}} \mathfrak{g}_\zeta(x + \alpha)\; dx = \frac{\zeta}{2\pi} \int_{-\frac{\pi}{\zeta}}^{\frac{\pi}{\zeta}} \mathfrak{g}_\zeta(x)\; dx, \quad \forall\, \zeta \in J \subset [0,\, \infty). \tag{1.10}$$

Successive application of the triangle inequality with the mean-value Theorem (1.5) over l together with (1.10) leads uniformly, with respect to m, to the asymptotic result

$$\lim_{\zeta \to \infty} |\mathfrak{A}_\zeta(m)| = 0, \; \forall\, m, \tag{1.11}$$

and by that the proof completes. ∎

Remark 1. Not all dilations of measurable, bounded and I- periodic piece-wise continuous functions can satisfy (1.5).

We have seen, e.g., that $\mathfrak{h}(\zeta x)$ does not satisfy (1.5), while $\mathfrak{g}(\zeta x) = \sin \zeta x$ or $\cos \zeta x$ obviously do. The rather peculiar result of the previous lemma, which applies to a wide class of piece-wise continuous periodic $\mathfrak{g}_\zeta(x)$ like, e.g., $\mathfrak{h}_\zeta(x)$ of (1.4), can be understood in the following way. Asymptotically at any instant x, the average value of $\mathfrak{g}_\zeta(x)$ over an infinitesimally small interval of x is zero and regardless of its spanned strip-like range. This feature is illuminated further when conceived in a distributional sense, as in the result that follows, where the "redundant" rapid oscillations are smeared out under the action of integration.

Theorem 1 (General kernel RLL)**.** *If* $\mathfrak{g}_\zeta(x)$ *is any* l*-periodic function that satisfies the conditions of Lemma 1, and if* $f(x) \in L(I)$, *then*

$$\lim_{\zeta\to\infty} \int_I f(x)\ \mathfrak{g}_\zeta(x)\ dx = 0. \tag{1.12}$$

Proof. Lebesgue integrability of $f(x)$ and dominance of $\mathfrak{g}_\zeta(x)$ according to (1.6) guarantee that the summable sequence $f(x)\,\mathfrak{g}_\zeta(x)$ converges pointwise and is dominated by some integrable function over I. It is possible, then to invoke Lebesgue's dominated convergence theorem and write

$$\lim_{\zeta\to\infty} \int_I f(x)\,\mathfrak{g}_\zeta(x)\ dx = \int_I f(x) \lim_{\zeta\to\infty} \mathfrak{g}_\zeta(x)\ dx. \tag{1.13}$$

Finally, make use of the result (1.7) of Lemma 1 in (1.13) to complete the proof. ■

One of the various practical applications of the R-L lemma on R is that, if $f \in L^1(R)$, then its Fourier transform $\hat{f}(\zeta)$ tends to zero as $\zeta \to \infty$. The space of all functions for which the integral (1.8) exists in the Lebesgue sense is just $L^1(R)$ together with the space of basic test functions $S(R) \subset L^1(R) \cap L^2(R)$. This integral exists as a limit-in-mean for all $f \in L^p(R)$ for $1 < p < 2$ and for the Bohr space, $O(R)$, of almost periodic functions [5], which is endowed with $\|f\|_{1;\,\infty}$. It is also defined in the distributional sense for the $S'(R)$ space of generalized functions of slow growth which are generated by elements of $S(R)$.

Definition 1. The $F(R)$ space is the set of all functions or distributions, $f(x)$, for which the integral (1.8) exists, i.e., $F(R) \supset S'(R) \cup O(R) \cup L^r(R)$, $1 < r < 2$.

Definition 2. Let $B(R)$ be the space of bounded functions over R, then $B_0(R) = L_0^\infty(R)$ is the set of all compactly supported bounded functions, i.e., the space endowed with the norm $\|f\|_B = \sup_{x \in I} f(x)$, $I \subset R$.

Definition 3. The $T_0(R)$ space is the union of $B_0(R)$ with the set of unbounded compactly supported functions of slow growth [6].

Double-parameterization of the R-L Lemma for aperiodic $f \in T_0(R) \cap F(R)$ is shown in [6] to be

$$\lim_{|\delta| \to \infty} \Gamma_{\delta;\, \sigma}[f] = \lim_{|\delta| \to \infty} \int_R f(x) H_{\delta, \sigma}(x)\, dx = 0;\ \forall\, \sigma \in R,$$

with

$$\Gamma_{\delta;\, \sigma}[\ ] = \int_R H_{\delta, \sigma}(x)\, [\ ]\, dx : L^2(R) \to \boldsymbol{C},$$

having a generalized real Gábor kernel,

$$H_{\delta, \sigma}(x) = A(\delta, \sigma)\, [\, \cos \sigma x + \sin \delta x\,],$$

with a distinctive property of a dilation parameter $\delta = \delta(m, \lambda) = 2\pi \frac{m}{\lambda}$, that is, coupled to a translation parameter $\sigma = \sigma(m, \lambda) = \delta + \frac{3}{2} \frac{\pi}{\lambda} = 2\pi \frac{m}{\lambda} + \frac{3}{2} \frac{\pi}{\lambda}$, via $m \in Z$ and $\lambda \in (R \setminus \{0\})$, and of a single-parameterized amplitude function $A(\delta, \sigma) = A(\lambda) = \lambda^{-1}$ having a kernel $H_{\delta, \sigma}(x)$, that is, quite different from the classical Gábor transform [4] kernel

$$G_{\delta, \sigma}(x) = \Lambda(x - \sigma)\, [\, \cos \delta x + i \sin \delta x\,]. \tag{1.14}$$

While $G_{\delta, \sigma}(x)$ is a complex function, with a separated translation, σ, from dilation, δ, σ in $H_{\delta, \sigma}(x)$ is not a conventional translation parameter in the Gábor or wavelet sense.

Lemma 2 (Extended RLL [6])**.** *For*

$$\Gamma_{m,\lambda} = \int_R q_{\lambda,m}(x)[\ ]\,dx,$$

with

$$q_{\lambda,m}(x) = \lambda^{-1}[\sin m\frac{2\pi}{\lambda}x + \cos\nu\frac{2\pi}{\lambda}x],$$

$$\nu = (4m+3)/4, \tag{1.15}$$

$$\lim_{m\to\infty}\Gamma_{m,\lambda}[f] = 0,\ \forall f \in T_0(R),$$

and

$$\forall\lambda \in (R \setminus \{0\}). \tag{1.16}$$

Furthermore, the integral that stands in Lemma 2 defines a function

$$F\dot{G}\, f(\alpha,\ m) := \int_R f(x)\,\mathfrak{q}_{\lambda,m}(x)\,dx = \langle f,\ \mathfrak{q}_{\lambda,m}\rangle,\ \lambda \neq 0,$$

that turns out to be a Fourier-Gábor hybrid finite-integral transform of $f \in T_0(R)$ with respect to $\mathfrak{q}$.

2. A Riemann-Lebesgue Lemma with Wavelet-Like Kernels

In actual fact, the kernel $\mathfrak{g}_\zeta(x)$ for the RLL does not necessarily have to be a periodic function, and can be replaced by wavelet-like functions $\mathfrak{W}_\zeta(x)$ with a compact support $\Delta = [u,\ v] \subseteq I = [a,\ b]$, say. One possible example of such a wavelet is

$$\mathfrak{W}_\zeta(x) = \begin{cases} A\phi_\zeta(x), & x \in \Delta, \\ 0, & \text{elsewhere}, \end{cases}$$

with $0 < A < \infty$ and $\phi_\zeta : [u,\ v] = \Delta \to [-1,\ 1],\ \forall\zeta = 2\pi/l$ satisfying the integrated symmetry condition

$$\int_{-\frac{\pi}{\zeta}}^{\frac{\pi}{\zeta}} \phi_\zeta(x)\, dx = 0, \quad \forall\, \zeta \in J \subset [0, \infty). \tag{2.1}$$

This clearly allows for the wavelet defining condition

$$\int_\Delta \mathfrak{W}_\zeta(x)\, dx = 0,\ \forall\, \zeta \in [0, \infty). \tag{2.2}$$

to be satisfied.

It is noteworthy that the condition (2.1) is asymptotically entirely different from (1.5) and that $\zeta = 2\pi / I$ is just an intrinsic frequency of the $\mathfrak{W}_\zeta(x)$ wavelet, which is a periodic.

Theorem 3 (Wavelet-like kernel RLL). *Let* $\mathfrak{W}_\zeta : [u, v] = \Delta \to A$ $[-1, 1],\ \forall\, \zeta,\ 0 < A < \infty,$ *be a real function, that is, compactly supported over* $\Delta = [u, v] \subseteq I = [a, b]$ *and having* $\phi_\zeta(x)$ *as a measurable periodic function satisfying* (2.1). *If* $f : [a, b] = I \to R,$ *is bounded* $\forall\, x \in I,$ *by a certain nondecreasing element of* $L(\Delta)$, *i.e.,* $f(x) < F(x) \in L[\Delta],$ *then*

$$\lim_{\zeta\to\infty} \int_I f(x)\,\mathfrak{W}_\zeta(x)\, dx = 0. \tag{2.3}$$

Proof. In satisfying (2.1) $\phi_\zeta(x)$ should be decomposable as

$$\phi_\zeta(x) = \phi_\zeta^+(x) - \phi_\zeta^-(x), \tag{2.4}$$

with $\phi_\zeta^+(x)$ and $\phi_\zeta^-(x) : [u, v] = \Delta \to [0, 1],\ \forall\, \zeta,$ satisfying

$$\lim_{\zeta\to\infty} \int_\Delta \phi_\zeta^+(x)\, dx = \lim_{\zeta\to\infty} \int_\Delta \phi_\zeta^-(x)\, dx = \Delta. \tag{2.5}$$

It can easily be graphically illustrated that for any $f(x)$ that is bounded over Δ, there exists a nondecreasing bounding $F(x)$, such that

$$\left| \int_u^V f(x)\,\phi_\zeta(x)\, dx \right| \le \left| \int_u^V F(x)\,\phi_\zeta(x)\, dx \right|. \tag{2.6}$$

We may consider in

$$\int_I f(x)\mathfrak{W}_\zeta(x)\,dx = \int_\Delta f(x)\mathfrak{W}_\zeta(x)\,dx = A\int_U^V f(x)\,\phi_\zeta(x)\,dx,$$

the satisfaction of (2.6) to write

$$\left|\int_I f(x)\mathfrak{W}_\zeta(x)\,dx\right| \le A\left|\int_U^V F(x)\phi_\zeta(x)\,dx\right|.$$

Clearly, then

$$\left|\lim_{\zeta\to\infty}\int_I f(x)\mathfrak{W}_\zeta(x)\,dx\right|$$

$$\le A\left|\left\{\lim_{\zeta\to\infty}\int_U^V F(x)\phi_\zeta^+(x)\,dx - \lim_{\zeta\to\infty}\int_U^V F(x)\phi_\zeta^-(x)\,dx\right\}\right|. \tag{2.7}$$

Now, let us study each of the limits in the right hand side of (2.7) in some detail, starting with

$$\lim_{\zeta\to\infty}\int_U^V F(x)\phi_\zeta^+(x)\,dx. \tag{2.8}$$

One can invoke for the integral in (2.8), Steffensen's inequality [2] to write

$$\int_{V-\epsilon(\zeta)}^V F(x)\,dx \le \int_U^V F(x)\phi_\zeta^+(x)\,dx \le \int_U^{U+\epsilon(\zeta)} F(x)\,dx, \tag{2.9}$$

in which

$$\epsilon(\zeta) = \int_U^V \phi_\zeta^+(x)\,dx.$$

Next, multiply all terms in (2.9) by -1 to reverse its sense

$$-\int_{V-\epsilon(\zeta)}^V F(x)\,dx \ge -\int_U^V F(x)\phi_\zeta^+(x)\,dx \ge -\int_U^{U+\epsilon(\zeta)} F(x)\,dx. \tag{2.10}$$

Add then $\int_U^V F(x)\,dx$ to every term in (2.10) to arrive at

$$\int_{\mathcal{U}}^{\mathcal{V}} F(x)\,dx - \int_{\mathcal{V}-\epsilon(\zeta)}^{\mathcal{V}} F(x)\,dx \geq \int_{a}^{b} F(x)\,\psi_{\zeta}^{+}(x)\,dx$$

$$\geq \int_{\mathcal{U}}^{\mathcal{V}} F(x)\,dx - \int_{\mathcal{U}}^{\mathcal{U}+\epsilon(\zeta)} F(x)\,dx, \quad (2.11)$$

where

$$\psi_{\zeta}^{+}(x) = 1 - \phi_{\zeta}^{+}(x). \quad (2.12)$$

This result can easily and elegantly be reduced to

$$\int_{\mathcal{U}}^{\mathcal{V}-\epsilon(\zeta)} F(x)\,dx \geq \int_{\mathcal{U}}^{\mathcal{V}} F(x)\,\psi_{\zeta}^{+}(x)\,dx \geq \int_{\mathcal{U}+\epsilon(\zeta)}^{\mathcal{V}} F(x)\,dx. \quad (2.13)$$

We move on to analyze further the asymptotic behavior of (2.13). According to (2.5),

$$\lim_{\zeta\to\infty} \epsilon(\zeta) = \Delta = \mathcal{V} - \mathcal{U},$$

and consequently

$$\int_{\mathcal{U}}^{\mathcal{U}} F(x)\,dx \geq \lim_{\zeta\to\infty} \int_{\mathcal{U}}^{\mathcal{V}} F(x)\,\psi_{\zeta}^{+}(x)\,dx \geq \int_{\mathcal{V}}^{\mathcal{V}} F(x)\,dx,$$

i.e.,

$$0 \geq \lim_{\zeta\to\infty} \int_{\mathcal{U}}^{\mathcal{V}} F(x)\,\psi_{\zeta}^{+}(x)\,dx \geq 0.$$

Make then use of (2.12) in

$$\lim_{\zeta\to\infty} \int_{\mathcal{U}}^{\mathcal{V}} F(x)\,\psi_{\zeta}^{+}(x)\,dx = 0,$$

to obtain

$$\lim_{\zeta\to\infty} \int_{\mathcal{U}}^{\mathcal{V}} F(x)\,\phi_{\zeta}^{+}(x)\,dx = \int_{\mathcal{U}}^{\mathcal{V}} F(x)\,dx. \quad (2.14)$$

One can now follow the same steps of analysis, while assuming

$$\psi_{\zeta}^{-}(x) = 1 - \phi_{\zeta}^{-}(x),$$

to demonstrate also that

$$\lim_{\zeta \to \infty} \int_U^V F(x) \phi_\zeta^-(x)\, dx = \int_U^V F(x)\, dx. \tag{2.15}$$

Finally, consider (2.14) and (2.15) back in (2.7) to reach the required result. ■

A remarkable feature of this RLL is that the relative position of the compact support Δ of its wavelet-like kernel $\mathfrak{W}_\zeta(x)$ over I is not of any significance.

3. The Symmetrized Integral γ and its Riemann-Lebesgue Property

The Riemann integral of a function $y = f(x)$ over an interval $I = [a, b] \subset R$ of the real line R,

$$\rho = \int_a^b f(x)\, dx, \tag{3.1}$$

is based on the projection of the curve of $f(x)$ on the x- axis $(y = 0)$ of the Cartesian plane.

Let us assume that the range of $f(x)$, that corresponds to $[a, b]$, is $[f(a), f(b)] = [c, d] \subset R$ in order to state the definition that follows.

Definition 4. The symmetrized integral of the same $f(x)$ function is

$$\gamma = \int_\alpha^\beta \eta(\tau)\, d\tau, \tag{3.2}$$

with $\eta(\tau) = z$, obtained from $f(x) = y$, by means of a linear transformation $\Gamma : R^2 \to R^2$, such that

$$\Gamma(x, y)^T = \mathfrak{C}(x, y)^T = (\tau, z)^T, \tag{3.3}$$

with $\mathfrak{C} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, a rotation matrix and

$$(\alpha, \beta)^T = \mathfrak{C}(a, b)^T + \frac{1}{\sqrt{2}}(c - b, d + a)^T,$$

an affine map.

Clearly, γ is another Riemann integral of $f(x)$, that is, based on the projection of the curve of $f(x)$, over the $[\alpha, \beta]$ interval, on the median $(y = x)$ of the first quadrant of the Cartesian plane. This takes place instead of projection of $f(x)$ on the $y = 0$ axis in the ρ-integral. The symmetrized γ-integral should, nontheless, by no means be confused with a changed variable from [7],

$$\rho = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))\, g'(t)\, dt,$$

for the same ρ of (3.1).

Lemma 3. *If $f(x)$ is monotonically increasing or decreasing differentiable function over $[a, b]$, then $\eta(\tau)$ is a one-to one map over $[\alpha, \beta]$.*

Proof. By contradiction, assume without loss of generality, that $f(x)$ is a step-wise discontinuous function

$$f(x) = \begin{cases} c, & a \le x < t, \\ d, & t < x \le b, \end{cases}$$

with a trajectory having the end points $P(a, c)$ and $S(b, d)$, satisfying $b < d < c$, containing a step-down discontinuity from $Q(t, c)$ to $R(t, d)$. Obviously, the segmented trajectory PS projects to $[a, b]$ on the x-axis and to $[\alpha, \beta]$ on the τ-axis. Q and R project to the same point $t \in [a, b]$, and to ζ and $\xi \in [\alpha, \beta]$. Finally, since $\xi < \zeta$ as one sweeps the trajectory from P to S, then $\eta(\tau)$ is geometrically not a one-to-one map over $[\alpha, \beta]$.

A monotonically varying differentiable function is free of discontinuities of the first kind, like the one at t; eliminating the reason for violating the one-to-one behavior of its $\eta(\tau)$. ■

Furthermore, it is trivially observable that the magnitude of a ρ-integral is independent of the sense (step-up or step-down) of a possible discontinuity in its $f(x)$. In contrast, the corresponding value of its γ-integral can significantly vary with a change in this sense. Therefore, the γ-integral is distinctively more informative for discontinuous functions than the pertaining ρ-integral.

Definition 5. Let $f(x)$ be a monotonically varying differentiable function over $[a, b]$, then an end point variation of $f(x)$ at $x = a$ is

$$\varsigma_a = \varsigma[a, f(a)] = \frac{1}{4}(a^2 + ac - c^2) = \frac{1}{4}[a - c(\sqrt{2} - 1)][a + c(\sqrt{2} + 1)].$$

Based on the fundamental theorem that follows, the previous ς_a number turns out to play the role of a generalized integrated "antiderivative" in the calculus of the $\rho - \gamma$ difference integral.

Theorem 4 (Differenced integral calculus)**.** *If* ρ *is the conventional integral of a monotonically varying differentiable function* $f(x)$ *over* $[a, b]$ *and* γ *is its pertaining symmetrized integral*, *then*

$$\gamma - \rho = \varsigma_b - \varsigma_a. \tag{3.4}$$

Proof. The proof is simply geometrical. Sketch an arbitrary monotonically increasing or decreasing $f(x)$, that may intersect with $y = x$ at a finite number of points, and assume without loss of generality, that (a, c) and (b, d) are above the $y = x$ line. Subtract then from ρ the area of the trapezoid with vertices $(a, 0)$, (a, a), (b, b) and $(b, 0)$, which equals $\frac{1}{2}(b^2 - a^2)$, and the projection triangle on $y = x$ from (a, c), which equals $\frac{1}{4}(c - a)^2$. Finally, add the projection triangle on $y = x$

from (b, d), which equals $\frac{1}{4}(d-b)^2$, to obtain γ. The result of the theorem follows by direct rearrangement of these terms. ■

This result is similar to, though different from, the fundamental theorem of calculus. It indicates remarkably that for monotonically varying differentiable $f(x)$ one needs to know or compute only one of the two integrals ρ or γ, then use this theorem to find the other one. Nevertheless, the γ- integral, which can be quite different from ρ, can serve either as an alternative or a complement or reasonable natural extension of the ρ- integral in some graphical or charting applications. Here as before, we shall consider the index set $J \subset [0, \infty)$ for $\zeta = 2\pi / l$ to be discrete or "continuous". Moreover, satisfaction of (1.12) in a γ- integration sense shall be conceived, in the theorem that follows, as possession of an R-L property by the pertaining integral.

Theorem 5 (Riemann-Lebesgue property for γ). *If* $f \in \gamma(I)$, *γ- integrable, over* $[a, b] = I \subset R$, *and if* $G_\zeta(\tau)$ *is a family* $(G_\zeta \mid \zeta \in J)$ *of bounded γ- integrable l-periodic functions* $\tau \mapsto G_\zeta(\tau) \in \gamma^2(l)$, *square γ- integrable over l, which satisfy*

(i) $$\frac{\zeta}{2\pi}\int_{-\frac{\pi}{\zeta}}^{\frac{\pi}{\zeta}} G_\zeta(\tau)\,d\tau = 0,\ \forall\, \zeta \in J \subset [0, \infty), \tag{3.5}$$

and

(ii) $$|G_\zeta(\tau)| < U(\tau) \in \gamma[\Omega],\ \forall\, \zeta \text{ and } \forall\, \tau \in \Omega, \tag{3.6}$$

then

$$\lim_{|\zeta| \to \infty} \int_\alpha^\beta \eta(\tau) \quad G_\zeta(\tau) \quad d\tau = 0. \tag{3.7}$$

Proof. By applying same arguments of Lemma 1 and Theorem 1 over the interval $\Omega = [\alpha, \beta]$. ■

This property happens to be a remarkable feature of the γ- integration, which distinguishes it from other extensions of the

Riemann integration, like the Henstock integration [7], which does not possess a Riemann-Lebesgue property. Indeed, it is known [3] that for each orthonormal system (in particular the trigonometric one $\mathfrak{g}_n(x) = (\sin nx$ or $\cos nx)$ on $I = [a, b]$, there is a Henstock integrable function $f : I \to R$ such that $\lim_{n\to\infty} \int_I f(x)\, \mathfrak{g}_n(x)\, d\tau = \infty$. Apparently, this is a violation of the Riemann-Lebesgue property, which takes place as a result of the pertaining fact: $\limsup_{n\to\infty} V_a^b\, \mathfrak{g}_n = \infty$.

Theorem 6 (Extended Riemann-Lebesgue property for γ)**.** *If* f *is* γ-*integrable over* $I = [a, b] \subset R$, *so as* $\eta \in T_0(\Omega)$, $\Omega = [\alpha, \beta]$, *then* $\lim_{m\to\infty} \int_\alpha^\beta \eta(\tau)\, q_{\lambda, m}(\tau)\, d\tau = 0$, $\forall \lambda \in (R \setminus \{0\})$.

Proof. Due to the singularity of the kernel, $q_{\lambda, m}(\tau) = \lambda^{-1}[\sin m \frac{2\pi}{\lambda} \tau + \cos \nu \frac{2\pi}{\lambda} \tau]$, $\nu = (4m + 3)/4$, in the $\mathfrak{J}(\lambda, m)$ integral,

$$\mathfrak{J}(\lambda, m) = \int_\alpha^\beta \eta(\tau)\, q_{\lambda, m}(\tau)\, d\tau = 0, \; \forall m \in Z,$$

$$\lambda \in (R \setminus \{0\}), \tag{3.8}$$

and for the sake of simplicity of arguments, we shall avoid in this proof the operational-heuristic approach used in [6] or the real analysis approach of Theorem 1. The approach, to be followed here instead, will be Riemann integration-based.

We start with considering the double parameterized integral $\mathfrak{J}(\lambda, m)$ with

$$\lambda q_{\lambda, m}(\tau) = [\sin m \frac{2\pi}{\lambda} \tau + \cos \nu \frac{2\pi}{\lambda} \tau] = [\mathrm{Im}\left(e^{im\frac{2\pi}{\lambda}\tau}\right) + \mathrm{Re}\left(e^{i\nu\frac{2\pi}{\lambda}\tau}\right)], \tag{3.9}$$

$\forall m \in Z$ and $\forall \lambda$. Consequently,

$$q_{\lambda, m}(\tau) = \lambda^{-1}[\mathrm{Im}\left(e^{im\frac{2\pi}{\lambda}\tau}\right) + \mathrm{Re}\left(e^{i\nu\frac{2\pi}{\lambda}\tau}\right)]. \tag{3.10}$$

In fact, γ- integrability of f over $[a, b]$ means, among other things, that $\eta(\tau)$ is piecewise continuous and bounded over Ω. Therefore, for any given $\varepsilon > 0$, the interval $[\alpha, \beta]$ of the integral $\Im(\lambda, m)$ can be divided into $n - 1$ subintervals in each of which $\eta(\tau)$ varies by less than 2ε. Then $\exists$ a set $\{\tau_k\}_{k=0}^{n}$ such that, $\alpha = \tau_0 < \tau_1 < \tau_2 < \tau_3 < \ldots\ldots\ldots\ldots\ldots < \tau_n = \beta$ with $|\eta(\tau) - \eta(\tau_k)| < \varepsilon$, $\forall\, \tau \in [\tau_{k-1}, \tau_k]$. Obviously ε can be easily made a zero, when $n \to \infty$.

Boundedness of $\eta(\tau)$ on Ω means that $\exists\, 0 < M < \infty$ such that

$$|\eta(\tau)| < M, \ \forall\, \tau \in \Omega. \tag{3.11}$$

Then

$$\Im(\lambda, m) = \sum_{k=1}^{n} \eta(\tau_k) \int_{\tau_{k-1}}^{\tau_k} q_{\lambda, m}(\tau)\, d\tau + \sum_{k=1}^{n} \int_{\tau_{k-1}}^{\tau_k} [\eta(\tau) - \eta(\tau_k)]\, q_{\lambda, m}(\tau)\, d\tau.$$

Obviously

$$|\Im(\lambda, m)| \leq \left| \sum_{k=1}^{n} \eta(\tau_k) \int_{\tau_{k-1}}^{\tau_k} q_{\lambda, m}(\tau)\, d\tau \right|$$

$$+ \left| \sum_{k=1}^{n} \int_{\tau_{k-1}}^{\tau_k} [\eta(\tau) - \eta(\tau_k)]\, q_{\lambda, m}(\tau)\, d\tau \right|,$$

which by virtue of the Cauchy-Schwartz inequality becomes

$$|\Im(\lambda, m)| \leq \sum_{k=1}^{n} |\eta(\tau_k)| \left| \int_{\tau_{k-1}}^{\tau_k} q_{\lambda, m}(\tau)\, d\tau \right|$$

$$+ \sum_{k=1}^{n} \left| \int_{\tau_{k-1}}^{\tau_k} [\eta(\tau) - \eta(\tau_k)]\, q_{\lambda, m}(\tau)\, d\tau \right|.$$

Further consideration of the analytic continuation of (3.10) in the above inequality makes it possible to write

$$|\mathfrak{J}(\lambda,\ m)| \leq \lambda^{-1}\sum_{k=1}^{n}|\eta(\tau_k)|\left|\int_{\tau_{k-1}}^{\tau_k}\left(e^{im\frac{2\pi}{\lambda}\tau}+e^{i\nu\frac{2\pi}{\lambda}\tau}\right)d\tau\right|$$

$$+\lambda^{-1}\sum_{k=1}^{n}\left|\int_{\tau_{k-1}}^{\tau_k}[\eta(\tau)-\eta(\tau_k)]\left(e^{im\frac{2\pi}{\lambda}\tau}+e^{i\nu\frac{2\pi}{\lambda}\tau}\right)d\tau\right|.$$

Now

$$\left|\int_{\tau_{k-1}}^{\tau_k}\left(e^{im\frac{2\pi}{\lambda}\tau}+e^{i\nu\frac{2\pi}{\lambda}\tau}\right)d\tau\right| = \left|\frac{e^{im\frac{2\pi}{\lambda}\tau_k}-e^{im\frac{2\pi}{\lambda}\tau_{k-1}}}{im\,\frac{2\pi}{\lambda}}+\frac{e^{i\nu\frac{2\pi}{\lambda}\tau_k}-e^{i\nu\frac{2\pi}{\lambda}\tau_{k-1}}}{i\nu\,\frac{2\pi}{\lambda}}\right|$$

$$\leq \left(\frac{1}{m}+\frac{1}{\nu}\right)\frac{\lambda}{\pi},$$

$$\left|\int_{\tau_{k-1}}^{\tau_k}[\eta(\tau)-\eta(\tau_k)]\left(e^{im\frac{2\pi}{\lambda}\tau}+e^{i\nu\frac{2\pi}{\lambda}\tau}\right)d\tau\right|$$

$$\leq 2\,\varepsilon(\tau_k-\tau_{k-1}).$$

Putting these together finally yields

$$\mathfrak{J}(\lambda,\ m) \leq M\left(\frac{1}{m}+\frac{1}{\nu}\right)\frac{n}{\pi}+2(\beta-\alpha)\frac{\varepsilon}{\lambda}.$$

Clearly, as $m \to \infty,\ \nu \to \infty$. Consequently, $\mathfrak{J}(\lambda,\ m)$ can be made, for any $\lambda \neq 0$, as small as we like by choosing ε small and/or when $|m| \to \infty$. Here the proof ends. ■

Note remarkably, however, that despite the similarity in proofs of the R-L property for the γ- or ρ-integrals of *f*, an R-L property for the γ-integral is not the same as the R-L property for the ρ-integral (satisfaction of the R-L Lemma) of the same *f*; as to be illustrated by the example that follows.

Example 1. Consider $y = f(x) = x + [u(x) - u(x-1)]$. This f is not integrable over $[0, \infty)$ and does not possess an R-L property for its ρ-integral : $\rho = \int_0^\infty f(x)\,dx \to \infty$, i.e.,

$$\lim_{\zeta \to \infty} \int_0^\infty f(x) \sin \zeta x \, dx \neq 0.$$

In contrast, for the associated γ- integral of *f*, we have: $[\alpha, \beta] = [\frac{1}{\sqrt{2}}, \infty)$,

$\Upsilon[f(x)] = \eta(\tau) = \frac{1}{\sqrt{2}}\left[u\left(\tau - \frac{1}{\sqrt{2}}\right) - u\left(\tau - \frac{3}{\sqrt{2}}\right)\right]$, $\gamma = \int_{\frac{1}{\sqrt{2}}}^{\infty} \eta(\tau) \, d\tau = 1$, and

the γ- integral of *f* possesses, distinctively, an R-L property, i.e.,

$$\lim_{\zeta \to \infty} \int_{\frac{1}{\sqrt{2}}}^{\infty} \eta(\tau) \sin \zeta \tau \, d\tau = 0.$$

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